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Non-commutative gravity from the AdS/CFT correspondence.

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Abstract: The exclusion principle of Maldacena and Strominger is seen to follow from deformed Heisenberg algebras associated with the chiral rings of S_N orbifold CFTs. These deformed algebras are related to quantum groups at roots of unity, and are interpreted as algebras of space-time field creation and annihilation operators. We also propose, as space-time origin of the stringy exclusion principle, that the $AdS_3 \times S^3$ space-time of the associated six-dimensional supergravity theory acquires, when quantum effects are taken into account, a non-commutative structure given by $\mathrm{SU}_q(1,1) \times \mathrm{SU}_q(2)$. Both remarks imply that finite N effects are captured by quantum groups $\mathrm{SL}_q(2)$ with $q = e^{i\pi/(N+1)}$. This implies that a proper framework for the theories in question is given by gravity on a non-commutative spacetime with a q-deformation of field oscillators. An interesting consequence of this framework is a holographic interpretation for a product structure in the space of all unitary representations of the non-compact quantum group $\mathrm{SU}_q(1,1)$ at roots of unity.

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Contents

1. Introduction

The AdS/CFT [1, 2, 3] correspondence gives a large class of examples where the existence of a well defined theory of quantum gravity follows from the existence of a dual conformal field theory. Often we have access to regimes where perturbative string theory is not valid. The usual stringy explanations for the finiteness of the quantum gravity, like an infinite tower of massive states, are therefore not directly applicable. We can ask what, from the spacetime point of view, are the mechanisms that lead to a finite quantum theory of gravity.

Numerous recent studies have given considerable insight into the nature of the AdS/CFT correspondence. Some of the novel features of the emerging spacetime theory can be seen in [4]–[5]. For investigation of many of the outstanding questions regarding 5D black holes related to a system of Q_1 D1 and Q_5 D5-branes the CFT of $S^N(X)$ (X is $K3$ or T^4), where $N = Q_1Q_5$, provides a particularly useful laboratory [6, 7, 8, 9]. These orbifold CFTs have played important and diverse roles in recent works, see for example [10, 11, 12].

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One of the most interesting phenomena found in these studies is a stringy exclusion principle [4], related to the unitarity of representations of the superconformal algebra in the 2D CFT with target space $S^N(X)$. The N parameter here is analogous to the N of SU(N) Yang Mills theory in the $AdS_5 \times S^5$ case. Here $1/N$ is the expansion parameter for the semiclassical gravitational expansion for 6D gravity on $AdS_3 \times S^3$. The question that we address in the present work concerns the spacetime explanation of this exclusion principle and its meaning at the level of supergravity. We argue that the classical $AdS_3 \times S^3$ spacetime should be understood as a non-commutative manifold $\mathrm{SU}_q(1,1) \times \mathrm{SU}_q(2)$, with q a root of unity, when quantum effects are taken into account. We also suggest that an identical deformation applies to the (supergravity) field oscillators themselves. Earlier, an equivalent role of q-oscillators at roots of unity was proposed in [13], in the context of two dimensional string theory [14]. We are now lead to suggest that this represents a general feature of the matrix-spacetime correspondence.

In general a systematic way to define deformations of spacetime is studied under the heading of non-commutative geometry [15]. Gauge theory on non-commutative spaces has found a natural role in D-brane physics recently [16]. There is also a large literature attempting to define gravity in the non-commutative setting (for a review of the successes and difficulties see for example [17]). The version of gravity that we propose from CFT has some elements of similarity with the above literature, but also some differences. One characteristic feature implied in the present (and also other) non-commutative representations is the appearance of a space-time uncertainty relation [18].

The plan of the paper is the following. In section 2, we derive several important algebraic properties of the chiral ring of $S^N(X)$. For concreteness we work with $X = T^4$ but the arguments are general. We describe in a simple setting how the implementation of the exclusion principle requires that the creation and annihilation operators of fields obey a modified Heisenberg algebra. The relevant Heisenberg algebras are found from the CFT dual. They are shown to have a representation in terms of q-oscillators [19] with $q = e^{i\pi/N+1}$. The q-oscillators are also related to $\mathrm{SL}_q(2)$ (Note that in discussions where the real form does not matter much we use this notation, otherwise we will distinguish between $\mathrm{SU}_q(2)$ and $\mathrm{SU}_q(1,1)$ and even $\mathrm{SL}_q(2,\mathbb{R})$). At large N, we have the ordinary Heisenberg algebra. The discussion in the first part of section 2 is restricted to a simple class of field operators which, in the CFT, are related to the untwisted sector. We proceed to give some details on the construction of twisted sector operators. The same qualitative properties and deformation of Heisenberg algebras also survive when we consider the operators from the twisted sector.

The final part of section 2 derives the detailed form of the exclusion principle on the generators of the chiral primaries. Although the general fact of the existence of the exclusion principle follows from the unitarity of the superconformal algebra, the detailed form of the exclusion principle coming from the orbifold CFT is not predicted by this argument. Some interesting new features are derived and turn out to be precisely captured by the idea of a quantum group symmetry, as elaborated in section 3. The results can be expressed simply in terms of a bound of the left and right SU(2) quantum numbers of the chiral primaries.

Section 3 deals with a spacetime interpetation of these results. It was shown in [7] that KK reduction of supergravity fields matches the spectrum of chiral primaries at large N. An ordinary KK reduction on S^3 produces an infinite number of representations of SU(2). The truncation of the number of unitary representations is familiar in going from the universal enveloping algebra U(SU(2)) to $\mathrm{U}_q\mathrm{SU}(2)$. The deformation of the algebra can be related to a deformation of the SU(2) manifold to a non-commutative manifold $\mathrm{SU}_q(2)$ [20]. In terms of the geometry of $\mathrm{SU}_q(2)$ the cutoff on the spectrum of unitary reps of $\mathrm{U}_q\mathrm{SU}(2)$ means that Fourier transformation on the deformed manifold involves a finite set of SU(2) representations. (In the rest of this paper, we will not always be careful to use separate notation for the q-deformed algebra and the dual q-deformed manifold, using $\mathrm{SU}_q(2)$ for both, since it will often be clear from the context which one is being refered to.) The detailed form of the CFT cutoffs, derived in section 2, on the chiral primaries takes the precise form expected from KK reduction on $\mathrm{SU}_q(2)$ with $q^{\pi i/(N+1)}$, rather than SU(2).

The next part of section 3 shows that not only $\mathrm{SU}(2) = S^3$ is deformed to $\mathrm{SU}_q(2)$ but that the $AdS_3 = \mathrm{SU}(1,1)$ is also deformed to $\mathrm{SU}_q(1,1)$, again with the same value of q. This allows a discussion of the cutoffs on the superalgebra descendants of chiral primaries. Since we are claiming that the CFT living on an ordinary commutative Riemann surface is equivalent to gravity on a non-commutative spacetime, consistency with holography requires ordinary Riemann surface to appear in the boundary of $\mathrm{SU}_q(1,1)$. A result of [21] allows us to prove that it does. This discussion of holography requires, not surprisingly, a consideration of operators which allow the creation of states of sufficiently large mass that they can change the background into one with black holes.

We observe that the q values obtained from the deformed Heisenberg algebra and from the above q-deformation of SU(2) are identical. The agreement of the q-parameters from two different kinds of physics leads to a speculation on algebraic structures associated to the chiral ring and their space-time interpretation. We then outline an approach, based on the ordinary WZW CFT/quantum group correspondence, to make the presence of non-commutative spacetime coordinates explicit. In the final part of section 3, we attempt to get a glimpse of the possibilities coming from generalizing the idea of q-deformed spacetime to other AdS backgrounds entering the AdS-CFT correspondence.

We conclude with a summary and outline possible strategies to improve the understanding of the relevance of gravity in the non-commutative setting to the elucidation of quantum effects in the AdS-CFT correspondence.

2. Algebraic properties of chiral ring of SCFT on $S^N(T^4)$

The spacetime fields in $AdS_3 \times S^3$ supergravity are related to chiral primary operators in the dual CFT. A detailed discussion of the spacetime physics requires a derivation of several properties of the chiral ring, directly from the CFT. The properties of interest fall into two broad categories. One concerns the nilpotence of the generators of the chiral ring and the second concerns the truncation in the number of generators. Sections 2.1 and 2.3 concern the former property. Section 2.4 concerns the latter. Section 2.2 contains some technical points necessary for the subsequent discussion.

2.1 Deformed oscillator algebra and $\mathrm{SL}_q(2)$

In the superconformal field with the symmetric product target space $S^N(T^4)$, consider the S_N invariant operator related to a $(1,1)$ form on T^4 of the form $\psi\bar{\psi}$.

$$\begin{aligned} \alpha_{-1} &= \sum_{i=1}^{N} \psi_i \bar{\psi}_i \\ \alpha_1 &= \sum_{i=1}^{N} \bar{\psi}_i^{\dagger} {\psi_i}^{\dagger} \, . \end{aligned} \tag{2.1}$$

The two fermions are each chosen from one of two complex fermions, from the left moving and the right moving part of the CFT respectively. The anti-commutators are

$$\begin{aligned} \{\psi_i, \psi_j^{\dagger}\} &= \delta_{ij} \\ \{\bar{\psi}_i, \bar{\psi}_j^{\dagger}\} &= \delta_{ij} . \end{aligned} \tag{2.2}$$

To make the connection with CFT we note the following

$$\psi = \psi_{-1/2}\, , \qquad \psi^{\dagger} = \psi^{*}_{1/2}\, , \qquad \bar{\psi} = \bar{\psi}_{-1/2}\, , \qquad \bar{\psi}^{\dagger} = \bar{\psi}^{*}_{1/2}\, . \tag{2.3}$$

In the CFT, the fields ψ, ψ^* are left movers (depending on z) while $\bar{\psi}, \bar{\psi}^*$ are right movers (depending on $\bar{z}$). Note that $\psi\bar{\psi}$ belongs to the (c,c) ring. ψ is annihilated by the SU(2) raising operator, $\psi^{\dagger}$ by the lowering operator. The product is not annihilated by either.

The commutator of the bosonic oscillators gives:

$$\left[\frac{1}{\sqrt{N}}\alpha_1, \frac{1}{\sqrt{N}}\alpha_{-1}\right] = 1 - \frac{1}{N}\sum_i \left(\psi_i\psi_i^{\dagger} + \bar{\psi}_i\bar{\psi}_i^{\dagger}\right) . \tag{2.4}$$

In the Hilbert space obtained by acting with the creation operators ψ on a vacuum annihilated by the $\psi^{\dagger}$ find that the second term acts as a number operator. Indeed defining

$$\begin{aligned} \hat{n} &= \frac{1}{2}\sum_i \left(\psi_i\psi_i^{\dagger} + \bar{\psi}_i\bar{\psi}_i^{\dagger}\right) . \\ [\hat{n}, \alpha_{\pm 1}] &= \pm\alpha_{\pm 1}\, , \end{aligned} \tag{2.5}$$

we can rewrite (2.4) as

$$[a_p, \bar{a}_p] = 1 - \frac{2\hat{n}}{N}\,. \tag{2.6}$$

This algebra has been studied as a deformation of the Heisenberg algebra and has been named the parafermion algebra (see [22] and earlier refs. there). One has the relation

$$\phi_p(\hat{n}) = \bar{a}_p a_p = \hat{n} - \frac{\hat{n}(\hat{n}-1)}{N}\,. \tag{2.7}$$

and also that $\phi_p(\hat{n}+1) = a_p\bar{a}_p$. We have shown that the chiral ring (and its SU(2) descendants) has relations of the same form as the parafermion algebra.

Another oscillator algebra which is related to the ring of chiral operators and their conjugates is the q-oscillator algebra, for q a root of unity. The relevance of the q-deformed oscillator algebra is easily guessed. Because the α_{-1} is constructed in terms of fermions, it satisfies $\alpha_{-1}^{N+1} = 0$. The usual Heisenberg algebra relation

$$[a, a^\dagger] = 1 \tag{2.8}$$

is not compatible with this nilpotence. A deformation which is compatible is the q-oscillator algebra

$$a_q a_q^\dagger - q^{-1} a_q^\dagger a_q = q^{\hat{n}}\,, \tag{2.9}$$

with $q = e^{i\pi/(N+1)}$. Indeed it can be checked that

$$a_q (a_q^\dagger)^{N+1} = -(a_q^\dagger)^{N+1} a_q\,. \tag{2.10}$$

So imposing the nilpotence is consistent with the relations of the algebra. The q-oscillator algebra can be related to $\mathrm{SL}_q(2)$ [19] by a map in which ${a_q}^2$ is related to X_-, ${a_q^\dagger}^2$ is related to X_+ and $\hat{n}$ is related to H.

A precise connection to the q-oscillator algebra, which we will develop, allows us to associate a Hopf algebra structure to the set of operators generated by the chiral operators and their conjugates. This will use results on the Hopf algebra structure of q-oscillators. Ref. [23] gives a co-product Δ, an antipode S, and an R-matrix for the q-oscillators. This allows us to associate all these structures to the algebra of chiral operators and their conjugates.

We can transform from the oscillators in (2.1) to the q-oscillators. A crucial role is played by the functions $\phi_p(\hat{n})$ and $\phi_q(\hat{n})$

$$\phi_p(\hat{n}) = \hat{n}\left(1 - \frac{1}{N}(\hat{n}-1)\right)\,, \qquad \phi_q(\hat{n}) = \frac{q^{\hat{n}} - q^{-\hat{n}}}{q - q^{-1}}\,. \tag{2.11}$$

To all orders in $1/N$, they can be used to map the parafermion oscillators and the q-oscillators to the ordinary Heisenberg algebra [22]. To map between parafermion oscillators and Heisenberg algebra, we use

$$a_p = a\sqrt{\frac{\phi_p(\hat{n})}{N}}\,. \tag{2.12}$$

This map is singular at finite N because $\phi_p(\hat{n} = N+1) = 0$. To map between q-oscillators and Heisenberg algebra we use

$$a_q = a\sqrt{\frac{\phi_q(\hat{n})}{N}}\,. \tag{2.13}$$

This is also well-defined in the large N expansion but becomes singular precisley at finite $\hat{n} = N+1$, if we choose $q = e^{i\pi/(N+1)}$. To map between parafermion algebra and q-oscillator algebra

$$a_p = a_q\sqrt{\frac{\phi_p(\hat{n})}{\phi_q(\hat{n})}}\,. \tag{2.14}$$

This map is well defined for values of $\hat{n}$ between 0 and $N+1$. These are the only values of interest because the $(cc^\dagger)$ ring is just the quotient of the parafermion algebra by the relation $a_p^{(N+1)} = 0, a_p^{\dagger\,N+1} = 0$. This quotient algebra can be mapped to the quotient of the q-algebra by the analogous relations

$$a_q^{(N+1)} = 0\,, \qquad a_q^{\dagger\,(N+1)} = 0\,.$$

Using this map to q-oscillators we can write down the co-product for the algebra over the complex numbers generated by the c and $c^\dagger$. For example

$$\Delta(a_p) = \Delta\left(a_q\sqrt{\frac{\phi_p(\hat{n})}{\phi_q(\hat{n})}}\right) = \Delta(a_q)\Delta\left(\sqrt{\frac{\phi_p(\hat{n})}{\phi_q(\hat{n})}}\right)\,, \tag{2.15}$$

where we used the algebra homomorphism property of the co-product.

To recapitulate, we have invoked q-oscillators as the underlying explanation of the nilpotence of the generators of the chiral ring. Note that this dervation of the cut-off on the number operators of the generators of the chiral ring has a similar flavour to the standard derivation based on the chiral superconformal algebra, but has important differences since the α are generically non-chiral objects from the worldsheet point of view and have a spacetime interpretation in terms of field oscillators. In general our CFT generates a sequence of oscillators labeled by the harmonic forms and an index labelling cycle lengths. A more careful look at the deformation of the algebra shows that the deformation terms appearing in the commutator of $\alpha_1^{1,1}$ with $\alpha_{-1}^{1,1}$ involve also the other operators which count the number of oscillators of other types. Requiring the positivity of $\langle\alpha_{-1}\alpha_1\rangle$ in states with various oscillator numbers imposes constraints on the values that these number operators can take at finite N. For example the terms involving the pure twist operators take the form given below:

$$\left[\alpha_1^{1,1}, \alpha_{-1}^{1,1}\right] = 1 - \frac{2}{N}\left(N_1 + \sum_l a_l|O_l'\rangle\langle O_l| + \cdots\right)\,, \tag{2.16}$$

where the a_l are some positive coefficients growing with the twist number l. We will not try to unravel these constraints here, which will in general involve not just chiral primaries but also their descendants. We will turn to a more direct determination of the constraints on the operators of greatest interest, the generators of the chiral ring, in section 2.4.

2.2 Twist operators

There are twist operators [24] for a boson which have conformal dimension given by:

$$\Delta(\sigma_{k,n}(X)) = \frac{k}{2n}\left(1 - \frac{k}{n}\right). \tag{2.17}$$

These satisfy the condition

$$\partial X(z)\sigma_{k,n}(X)(0) = z^{-(k/n-1)}\tau_{k,n} + \cdots \tag{2.18}$$

so that under transport around this twist operator we have the monodromy

$$X \to e^{2ik\pi/n} X\,. \tag{2.19}$$

For fermions, we have $\sigma_{k,n}(\psi)$, which have dimension

$$\Delta(\sigma_{k,n}(\psi)) = \frac{k^2}{2n^2} \tag{2.20}$$

and satisfy

$$\psi_{(k)}(z)\sigma_{k,n}(\psi) = z^{k/n}\tau + \cdots \tag{2.21}$$

After bosonization,

$$\psi_{(k)} = e^{i\phi_{(k)}}\,, \qquad \sigma_{k,n}(\psi_{(k)}) = e^{i\frac{k}{n}\phi_{(k)}}\,. \tag{2.22}$$

We have formed linear combinations

$$\psi_{(k)} = \psi_1 + \omega^k\psi_2 + \cdots + \omega^{kn}\psi_n\,, \tag{2.23}$$

which transform by a phase ω^k under the action of the n-cycle permutation. The boson $\phi_{(k)}$ bosonizes the pair $\psi_{(k)}$ and $\bar{\psi}_{(k)}$. Take the operator

$$\sigma_n(\psi) = \prod_{k=1}^{n-1} e^{i\frac{k}{n}\phi_{(k)}}\,. \tag{2.24}$$

It has conformal weight $n/6 + 1/12n - 1/4$.

For the torus T^4 we have two complex bosons and two complex fermions so the operators $\alpha_{-n}(X^i, \psi^i)$ are built by combining twist operators for the bosons and fermions, with the index i running from 1 to 2:

$$\alpha_{-n}(X^i, \psi^i) = \prod_{i,k} e^{i\frac{k}{n}\phi^i_{(k)}} \prod_{i,k} \sigma_{k,n}(X^i_{(k)})\,. \tag{2.25}$$

For two complex fermions, the conformal weight is $n/3 + 1/6n - 1/2$. For the bosons we have $n/6 - 1/6n$. Adding up the dimensions of the bosonic and fermionic twist operators, we get the weight $(n-1)/2$. The operator $e^{i\frac{k}{n}\phi_{(k)}}$ has a U(1) charge of k/n. After summing over k we get a U(1) charge of $(n-1)$. Using the expressions for the left moving SU(2) currents (and the analogous ones for the right-moving ones) we can easily check that the above operator is annihilated by J_0^+:

$$J^0 = \psi^1(\psi^1)^* + \psi^2(\psi^2)^*\,, \qquad J^+ = \psi^1\psi^2\,, \qquad J^- = (\psi^1)^*(\psi^2)^*\,. \tag{2.26}$$

This confirms we have a chiral primary obeying as it should, $L_0 = 1/2 J_0$ [25].

We have constructed above the twist operator for a Z_n theory. In the case of S_N we just sum the Z_n twist operator over all combinations of n variables

$$\alpha_{-n}^{(0,0)}(z,\bar z) = \sum_{i_1\cdots i_n} \sigma(X^i_{i_1\cdots i_n}, \psi^i_{i_1\cdots i_n})\,. \tag{2.27}$$

The indices $i_1, \ldots, i_n$ are n distinct numbers between 1 to N. The most general chiral primary can be written in terms of products of operators $\alpha_{-n}^{(p,q)}$ associated with a (p,q) form $\omega^{(p,q)}(X^i, \psi^i)$ which is

$$\alpha_{-n}^{(p,q)} = \sum_{i_1,\ldots,i_n} \omega^{(p,q)}(X^i_{i_1} + \cdots + X^i_{i_n}, \psi^i_{i_1} + \cdots + \psi^i_{i_n})\sigma(X^i_{i_1\cdots i_n}, \psi^i_{i_1\cdots i_n})\,. \tag{2.28}$$

Schematically we will write, using a factorization of the Z_n twist operators in terms of chiral and anti-chiral operators,

$$\alpha_{-n}(z,\bar z) = \sum_{I_n} O^{I_n}(z)\bar O^{I_n}(\bar z)\,, \tag{2.29}$$

where O^{I_n} depends on X and ψ fields labeled by I_n, which equals a set of indices $(i_1, \cdots i_n)$. The indices $i_1, i_2 \cdots i_n$, run over n *distinct* numbers between 1 to N. This acts on the vacuum to give a state

$$\lim_{z,\bar z\to 0} \alpha_{-n}(z,\bar z)|0\rangle = \int \frac{dz}{z}\frac{d\bar z}{\bar z} O^{I_n}(z)\bar O^{I_n}(\bar z)|0\rangle\,. \tag{2.30}$$

We can define the z-independent operator

$$\alpha_{-n} = \sum_{I_n} \int \frac{dz}{z} O^{I_n}(z) \int \frac{d\bar z}{\bar z}\bar O^{I_n}(\bar z)\,. \tag{2.31}$$

The dual operator is naturally defined by taking the standard CFT dual

$$\alpha_n = \int \frac{dz}{z}\frac{d\bar z}{\bar z} z^{-2\Delta_n}\bar z^{-2\bar\Delta_n}\alpha_{-n}(z,\bar z)\,. \tag{2.32}$$

This changes the sign of the oscillator number associated with a conformal field. Since the expansion of a conformal field O^{I_n} of weight Δ looks like $\sum_m O^{I_n}_m z^{-m-\Delta_n}$, the contour integral will extract $O^{I_n}_{-\Delta_n}$. Operators $O^{I_n}_{-\Delta_n - m}$ are descendants of chiral primaries for $m > 0$.

2.3 Deformed Heisenberg algebras of the twist operators

In the previous subsection, we discussed deformed Heisenberg algebras associated with operators in the untwisted sector of the CFT. These deformed algebras led to the nilpotence of the generators of the chiral ring. They may also be expected to lead to a finiteness in the number of generators of the chiral ring. We will see that the same qualitative properties can be deduced from the deformed Heisenberg algebras associated with oscillators coming from the twisted sector. We will use here some of these facts about twist operators derived in the previous subsection.

We can write the relations of the Heisenberg algebra, taking n even:

$$\begin{aligned}
[\alpha_n, \alpha_{-n}] &= \sum_{I_n, I'_n} [\bar{O}^{\dagger I_n} O^{\dagger I_n}, O^{I'_n} \bar{O}^{I'_n}] \\
&= \{O^{\dagger I_n}, O^{I'_n}\}\{\bar{O}^{\dagger I_n}, \bar{O}^{I'_n}\} - \{O^{\dagger I_n}, O^{I'_n}\}\bar{O}^{I'_n}\bar{O}^{\dagger I_n} - \{\bar{O}^{\dagger I_n}, \bar{O}^{I'_n}\}O^{I'_n}O^{\dagger I_n} \\
&= \delta_{I_n I'_n}\delta_{I_n I'_n} - \delta_{I_n I'_n}\bar{O}^{I'_n}\bar{O}^{\dagger I_n} - \delta_{I_n I'_n} O^{I'_n} O^{\dagger I_n} \\
&= C_{n,N} - (\bar{O}^{I_n}\bar{O}^{\dagger I_n} + O^{I_n} O^{\dagger I_n}) \,.
\end{aligned} \tag{2.33}$$

$C_{n,N}$ is the (positive) number of elements in the conjugacy class in S_N characterized by one non-trivial cycle of length n. To leading order in large N this coefficient behaves like N^n. The separate purely chiral and purely anti-chiral commutators are computed using contour integrals.

$$\begin{aligned}
\{O^{I_n}, O^{\dagger I'_n}\} &= -\int dw w^{2\Delta_n - 1} \int dz \left(1 + \frac{(z-w)}{w}\right)^{-1} O^{I_n}(z) O^{*I'_n}(w) \\
&= \delta_{I_n I'_n} \,.
\end{aligned} \tag{2.34}$$

After redefining the normalization of these oscillators by dividing by this order of a conjugacy class we can get 1 as the leading term. The result for the deformed Heisenberg algebra is

$$[\alpha_n, \alpha_{-n}] = 1 - \frac{1}{C_{n,N}} \left(\bar{O}^{I_n}\bar{O}^{\dagger^{I_n}} + O^{I_n} O^{I_n \dagger}\right) \tag{2.35}$$

The operator $\bar{O}^{I_n}\bar{O}^{\dagger^{I_n}} + O^{I_n} O^{I_n \dagger}$ is positive and appears with a minus sign in front of it (for operators α_{-n} with n odd the sign is different and cutoffs could be derived by arguments using more detailed properties of these operators). It can be expressed as a sum, with positive coefficients, of number operators for the various oscillators in the theory. This can be done along the lines of the calculation in section 2. The positivity of the coefficients in front of the number operators follows from the fact that they are related to the numbers which appear in the class multiplication algebra of the symmetric group. We pause to explain what we mean by this. Let T_a be a conjugacy class in the symmetric group S_N. Define in the group algebra of S_N the sum of all the elements in T_a, and denote it, by a slight abuse of notation, T_a. For two conjugacy classes T_a and T_b we have a product which can be expanded

$$T_a T_b = C^c_{ab} T_c\,. \tag{2.36}$$

The coefficients C^c_{ab} are positive. Thus there are various constraints on the the values of the number operators coming from the requirement that $< aa^\dagger >$ is positive. This puts restrictions on the values that the number operators can take on the states obtained by acting with the chiral primary operators on the vacuum.

It is very interesting that the commutators between oscillators corresponding to permutations of different cycle lengths are also non-trivial, although they vanish at large N.

$$[\alpha_{l_1}, \alpha_{-l_2}] = a_1 \alpha_{-(l_2-l_1+1)} + a_2 \alpha_{-(k_1,k_2\cdots)} + \cdots \tag{2.37}$$

We have on the right hand side oscillators corresponding to permutations for which the sum of cycle lengths adds up to $l_2 - l_1$.

Relations of the type (2.37) imply that underlying this system there should be a description given by a Hamiltonian of the form

$$H = H_0 + \sum_{ijk} C_{ijk} \alpha_i \alpha_j \alpha_k + \cdots \tag{2.38}$$

The H_0 is a sum related to the deformed Heisenberg algebras associated to each individual CFT chiral primary operator (and descendants). The coefficients C_{ijk} are be determined by group theory together with the intersection form on the manifold X. More precisely we recall that the index i is really a double index $(i_1 i_2)$, i_1 labeling cycle lengths and i_2 labeling forms on the manifold X. The coefficients C_{ijk} may be expected to decompose into a product $C_{i_1 j_1 k_1}$ which are struture constants of symmetric group multiplication, and $C_{i_2 j_2 k_2}$ which are intersection numbers coming from the manifold X. Interesting questions relate to the inclusion of non-chiral primaries in (2.38).

2.4 Generators of the chiral ring

The single particle states in gravity are identified with the generators of the chiral ring of the CFT. Multiparticle states are associated with chiral primary operators which are products of the generators [7]. It will be important to clarify the nature of the cutoffs on generators of the chiral ring which comes from the orbifold CFT. For concreteness we will discuss these relations in the context of the CFT of $S^N(T^4)$. For this purpose we will write down some explicit formulae for the chiral primary operators associated with the twisted sectors of the S_N orbifold theory. We argue that the simple pattern that emerges from such a discussion extends to the CFT for $S^N(K3)$.

Among the generators of the chiral primaries are $\alpha^{0,0}_{-n}$. The operator $\alpha^{0,0}_{-1}$ operator corresponds to the vacuum state in CFT, so it does not correspond to a particle excitation in gravity. In an S_N orbifold CFT the index n is clearly cutoff at N. The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ quantum numbers $(2J_L, 2J_R)$ of $\alpha^{(0,0)}_{-N}$ are $(N-1, N-1)$.

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The CFT also contains operators $\alpha_{-n}^{p,q}$ for any (p,q) and any n extending up to N. However some of these have the interesting property that they can be written in terms of products of generators. It will turn out that these relations have a remarkably simple consequence. The cutoffs can be described by saying that, for the generators, we have a bound

$$\max\,(2J_L, 2J_R) = (N-1)\,. \tag{2.39}$$

Note that this cutoff is stronger than the one based on the unitarity of the superconformal algebra [25] which says that $\max\,(2J_L, 2J_R) = c/3 = 2N$. In the next section we will explain that the precise cutoff (2.39) follows from a simple spacetime argument based on the picture that quantum effects result in a q-deformation of spacetime, with $q = e^{i\pi/(N+1)}$.

The first set of relevant relations takes the form

$$\alpha_{-N}^{(p,q)} = \alpha_{-1}^{(p,q)}\alpha_{-N}^{(0,0)}\,. \tag{2.40}$$

For $(p=1, q=0)$ and $(p=0, q=1)$, this follows trivially from the definitions:

$$\begin{aligned}
\alpha_{-1}^{(1,0)} &= \sum_{i=1}^{N}\psi_i^a \\
\alpha_{-N}^{(1,0)} &= \alpha_{-N}(X^i,\psi^i)\sum_{i=1}^{N}\psi_i^a \\
&= \alpha_{-N}^{(0,0)}\alpha_{-1}^{(1,0)}\,.
\end{aligned} \tag{2.41}$$

The next case will illustrate the form of the argument which can be used in all the cases in (2.40).

$$\begin{aligned}
\alpha_{-N}^{(1,1)} &= \left(\sum_{i=1}^{N}\psi_i^a\bar{\psi}_i^{\bar{b}}\right)\alpha_{-N}^{(0,0)} \\
&= \left(\sum_{k=0}^{N-1}\psi_{(k)}^a\bar{\psi}_{(N-k)}^{\bar{b}}\right)\alpha_{-N}^{(0,0)}\,,
\end{aligned} \tag{2.42}$$

where the objects with simple Z_N transformation properties $\psi_{(k)}$ have been defined in (2.23). Now we have to recall that the product that defines the relations between the generators of the chiral ring involves taking the OPE of the CFT fields and analysing the leading z-independent term. The $k=0$ term certainly gives a contribution to the leading constant term. However the other terms give terms that vanish in the $z=0$ limit. But the $k=0$ term is immediately identified with the first line of (2.42). The case of $K3$ involves a slightly more complicated form for these operators. We will have, in general, some function of X appearing in the form $f(X)\psi\bar{\psi}$. The similar decomposition into Z_N invariants will have a sum $\sum_{k_1,k_2,k_3} f_{(k_1)}\psi_{(k_1)}\psi_{(k_3)}$, where $k_1+k_2+k_3 = 0$ (mod N). For a non-zero leading term in the OPE we again have $k_1=k_2=k_3=0$.

Now consider the cases either p or q is equal to 2, but they are not both equal to 2. In these cases we show directly here that $\alpha^{(p,q)}_{-(N-1)}$ can be written in terms of products of other generators. The proof in all these cases is essentially of the same form, so we will illustrate in the case of $\alpha^{(2,1)}_{-(N-1)}$. The candidate operators which can give $\alpha^{(2,1)}_{-(N-1)}$ are

$$\begin{aligned} &\alpha^{(2,0)}_{-(N-1)}\alpha^{(0,1)}_{(-1)} \\ &\alpha^{(1,0)}_{-(N-1)}\alpha^{(1,0)}_{(-1)}\alpha^{(0,1)}_{(-1)} \\ &\alpha^{(0,0)}_{-(N-1)}\alpha^{(1,0)}_{(-1)}\alpha^{(1,0)}_{(-1)}\alpha^{(0,1)}_{(-1)} \end{aligned} \tag{2.43}$$

The set of operators that can appear is restricted by conservation of charge $(2J_L, 2J_R)$, and by the structure constants of symmetric group multiplication. As an illustration of the latter constraint, the first possibility is allowed because the multiplication of the conjugacy class containing one cycle of length $(N-1)$ associated with $\alpha^{(2,0)}_{-(N-1)}$ with the identity permutation associated with $\alpha^{(0,1)}_{(-1)}$ can give the first conjugacy class again which is associated with $\alpha^{(2,1)}_{-(N-1)}$.

To simplify notation in this section denote the operator $\sigma(X^i_{i_1\cdots i_n}, \psi^i_{i_1\cdots i_n})$ by $\sigma_{i_1\cdots i_n}$. Consider the product

$$\begin{aligned} \alpha^{(1,0)}_{-(N-1)}\alpha^{(1,0)}_{-1}\alpha^{(0,1)}_{-1} &= \sum_{(i_1,\cdots i_{N-1})} \sigma_{i_1\cdots i_{N-1}} \left(\psi^1_{i_1} + \psi^1_{i_2} + \cdots + \psi^1_{i_{N-1}}\right) \sum_j \psi^2_j \sum_k \bar\psi^2_k \\ &= \sum_{(i_1\cdots i_{N-1})} \sigma_{i_1\cdots i_{N-1}} \left(\psi^1_{i_1} + \psi^1_{i_2} + \cdots + \psi^1_{i_{N-1}}\right) \times \\ &\qquad\qquad \times \left(\psi^2_{i_1} + \cdots + \psi^2_{i_{N-1}}\right)\left(\bar\psi^1_{i_1} + \cdots + \bar\psi^1_{i_{N-1}}\right) + \\ &\quad + \sigma_{i_1\cdots i_{N-1}} \left(\psi^1_{i_1} + \cdots + \psi^1_{i_{N-1}}\right)\left(\psi^2_{i_1} + \cdots + \psi^2_{i_{N-1}}\right)\bar\psi^1_{i_N} + \\ &\quad + \sigma_{i_1\cdots i_{N-1}} \left(\psi^1_{i_1} + \cdots + \psi^1_{i_{N-1}}\right)\psi^2_{i_N}\left(\bar\psi^1_{i_1} + \cdots + \bar\psi^1_{i_{N-1}}\right) + \\ &\quad + \sigma_{i_1\cdots i_{N-1}} \left(\psi^1_{i_1} + \cdots + \psi^1_{i_{N-1}}\right)\psi^2_{i_N}\bar\psi^1_{i_N} \\ &= \alpha^{2,1}_{-(N-1)} + \left(-\alpha^{(2,1)}_{-(N-1)} + \alpha^{(2,0)}_{N-1}\alpha^{(0,1)}_{-1}\right) \times \\ &\quad \times \left(-\alpha^{(2,1)}_{-(N-1)} + \alpha^{(1,1)}_{-(N-1)}\alpha^{(1,0)}_{-1}\right) \times \\ &\quad \times \left(-\alpha^{(2,1)}_{-(N-1)} + \alpha^{(1,0)}_{-(N-1)}\alpha^{(1,1)}_{-1}\right). \end{aligned} \tag{2.44}$$

So all the terms which are allowed by S_N symmetry together with the U(1) charge conservation appear in the above equation.

This can be rewritten as

$$\begin{aligned} 2\alpha^{(2,1)}_{-(N-1)} &= \alpha^{(1,0)}_{-(N-1)}\alpha^{(1,0)}_{-1}\alpha^{(0,1)}_{-1} + \alpha^{(2,0)}_{-(N-1)}\alpha^{(0,1)}_{-1} + \\ &\quad + \alpha^{(1,1)}_{-(N-1)}\alpha^{(1,0)}_{-1} + \alpha^{(1,0)}_{-(N-1)}\alpha^{(1,1)}_{-1}. \end{aligned} \tag{2.45}$$

Finally we outline the proof of an equation expressing the operator $\alpha^{(2,2)}_{-(N-1)}$ in terms of other operators. We can start with a product of the form $\alpha^{(0,0)}_{-N}\alpha^{(0,0)}_{-2}$. Charge conservation and the structure of the symmetric group allows products of the form

$$\begin{aligned}&\alpha^{(2,2)}_{-(N-1)}\\&\alpha^{(p_1,q_1)}_{-(N-k)}\alpha^{(p_2,q_2)}_{-k}\,,\end{aligned}\tag{2.46}$$

where k is an odd number. A term like $\alpha^{(1,1)}_{-(N)}$ is allowed by left and right U(1) charge conservation, but it is not allowed by the multiplication law of the symmetric group, since the product of a permutation with a single non-trivial cycle of length $(N-1)$ and another with length 2 is even or odd depending on the sign of $(-1)^N$ whereas the parity of the permutation with cycle of length N is $(-1)^{(N-1)}$. Thus we may expect a relation with the product $\alpha^{0,0}_{-N}\alpha^{0,0}_{-2}$ on the left and on the right hand side a term proportional to $\alpha^{(2,2)}_{-(N-1)}$ and other terms proportional to products of α. No other term with a single α can appear. Terms corresponding to smaller cycles would require associated forms of degrees which exceed 2. As a result, $\alpha^{(2,2)}_{-(N-1)}$ can be written in terms of products of other α's, so the highest n allowed for $\alpha^{(2,2)}_{-n}$ among the generators of the chiral ring has $n = N-2$ and charges $2J_L = 2J_R = N-1$.

3. Spacetime interpretation of the exclusion principle

We now proceed to discuss the spacetime interpretation of the exclusion principle. The first main result in this section is that the detailed form of the cutoffs on the generators of the chiral ring has a simple interpretation in terms of our proposal that spacetime is deformed from $AdS_3\times S^3$ to $\mathrm{SU}_q(1,1)\times\mathrm{SU}_q(2)$. In section 3.1, we review some properties of $\mathrm{SU}_q(2)$ which follow from the consideration of study of unitary representations of $\mathrm{U}_q\mathrm{SU}(2)$. We then identify the q parameter associated with the deformed geometry, by comparing to the discussion in the last part of section 2. In section 3.2, we present arguments in favour of the deformation of the non-compact part of spacetime (in addition to the deformation of the compact part which played a direct role in the discussion of the chiral primaries). In section 3.3, we show that the deformation to $\mathrm{SU}_q(1,1)$ plays an important role in the discussion of the cutoffs on descendants of chiral primaries. Section 3.4 discusses holography in the q-deformed context. In section 3.5 we observe the equality of the geometrical q-parameter with the one obtained from deformed Heisenberg algebras in section 2, and we discuss its implications. In section 3.6 we outline some steps towards an explicit derivation of a non-commutative space-time coordinates from the orbifold CFT. In the final part we begin a very preliminary discussion of other AdS backgrounds.

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3.1 Non-commutative S^3 and the cutoff on chiral primaries

The above cutoffs on the left and and right SU(2) quantum numbers of the generators of the chiral ring have an interpretation in terms of KK reduction on a non-commutative deformation of $S^3 = \mathrm{SU}(2)$ to $\mathrm{SU}_q(2)$ with $q = e^{i\pi/(N+1)}$. We have the oscillators $\alpha_{-l}^{(p,q)}$ associated with single particle states of supergravity on $AdS_3 \times S^3$ carrying SU(2) spins $2J_L = l-1+p$ and $2J_R = l-1+q$. The KK reduction on S^3 gives states characterized by the reps of $\mathrm{SU}(2) = S^3$ because functions on SU(2) can be expanded in terms of matrix elements of representations of the group. The manifold S^3 being isomorphic to the group SU(2) admits a natural deformation to $\mathrm{SU}(2)_q$ [20] which preserves many group structures. Such a manifold admits $\mathrm{SU}_q(2) \times \mathrm{SU}(2)_q$ symmetry.

The universal enveloping algebra $\mathrm{U}_q\mathrm{SL}(2,\mathbb{C})$ is generated by H, X_+, X_- with the relations

$$\begin{aligned} [H, X_\pm] &= \pm X_\pm \\ [X_+, X_-] &= \frac{(q^{2H} - q^{-2H})}{q - q^{-1}} \,. \end{aligned} \tag{3.1}$$

The algebra is also equipped with a co-product $\Delta : A \to A \otimes A$, where $A = \mathrm{U}_q\mathrm{SL}(2,\mathbb{C})$. As an example

$$\Delta(X_+) = X_+ \otimes q^H + q^{-H} \otimes X_+ \,. \tag{3.2}$$

Note that the coproduct is not invariant under permutation of the first and second copies of the algebra. The two co-products are related by the R-matrix. This non-cocommutativity leads to the fact that the manifolds constructed from these algebras have non-commutative spacetime coordinates.

Specifying a real form of the Lie algebra, e.g. a compact form $\mathrm{SU}(2)_q$ or a non-compact form $\mathrm{SU}(1,1)_q$ involves the choice of an involution $*$ which is a map from the algebra to itself, satisfying

$$*(ab) = (*a)(*b) \,. \tag{3.3}$$

The q-algebra is also equipped with a coproduct $\Delta : A \to A \otimes A$. One also requires a compatibility between the conjugation and the coproduct which can take one of two forms.

$$*\Delta = \Delta\, * \tag{3.4}$$

$$\Delta = \Delta' * \tag{3.5}$$

If we require (3.4) the SU(2) involution exists for real q but not $|q| = 1$. If we require (3.5), the compact involution exists for roots of unity. We will discuss involutions which go to those related to SU(1,1) in the limit $q \to 1$ in section 4.

Given a definition of conjugation, compatible with the q-deformed algebra we can discuss unitarity of representations. The set of unitary highest weight reps of $\mathrm{SU}(2)_q$ for $q = e^{i\pi/(k+2)}$ is truncated to those having $2j \leq k$. The algebra of functions on the non-commutative manifold $\mathrm{SU}(2)_q$ at roots of unity is given by the matrix elements of a finite set of representations.

The existence of $\mathrm{SU}(2)_q \times \mathrm{SU}(2)_q$ symmetry associated with the q-sphere implies that KK reduction on the non-commutative sphere should correspond to a representations which have $(2J_L, 2J_R)$ which are both bounded by k. Equivalently $\max(2J_L, 2J_R) \leq k$. This is precisely the form of the cutoffs we get from the above discussion, with a value of $q = e^{i\pi/(N+1)}$.

In this way we are lead to a suggestion that some of the finite N constraints in the $AdS_3 \times S^3$ background can be captured by Kaluza-Klein reduction on a non-commutative 3-sphere $\mathrm{SU}(2)_q$.

3.2 Non-commutative AdS_3 and $\mathrm{SU}_q(1,1)$

We have argued that finite N effects lead to a deformation of the classical S^3 of spacetime to a q-deformed S^3. We will argue here that the finite N effects also deform the AdS_3 part of spacetime into a non-commutative manifold. The definition of the non-compact quantum group has a number of subtleties. We use the proposed relation between the non-compact quantum group and the CFT to learn something about the correct properties of the non-compact quantum group.

In the classical case, KK reduction of scalars on the sphere S^3 leads to states which correspond to the chiral primaries which are finite dimensional highest weight reps of $\mathrm{SU}(2)$ and infinite-dimensional highest weight reps of $\mathrm{SU}(1,1)$. The highest weight h of the $\mathrm{SU}(1,1)$ is related to the highest weight j of the $\mathrm{SU}(2)$ by $h = j/2$. This follows, from the spacetime point of view, by writing down a sum of laplaceans on the sphere and the AdS space, and taking into account the mixing between the scalars and some modes of the antisymmetric tensor fields. The net effect is to obtain a relation of the form

$$4h(h-1) = j(j-2)\,, \tag{3.6}$$

which allows a solution $h = j/2$. We expect that the KK reduction of the S_q^3 involves the q-Casimir rather than the ordinary Casimir. A condition of the form $h = j/2$ present in CFT, will only follow, if the $\mathrm{SU}(1,1)$ is also q-deformed.

This gives an argument that the $AdS3$ $(= \mathrm{SU}(1,1))$ space is also q-deformed. Deformations of the non-compact real forms are significantly more subtle than the compact form. For example, one non-compact form of [26] exists for real $q \neq 1$ at the level of Hopf algebra but does not have the right properties to lead to a quantum group as a non-commutative manifold [27]. This subtlety, fortunately does not concern us since we have a q-deformed $\mathrm{Sl}(2)$ for q equaling a root of unity.

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Deforming the algebra of bounded functions on the non-compact group will involve replacing a set of classical unitary representations of SU$(1,1)$ with some class of unitary representations of SU$(1,1)_q$. Typically in quantum group-CFT relations the structure of the quantum group representations is similar to those appearing in the CFT. In this case the conformal field theory contains SU$(1,1)$ representations which have a highest weight. So we need a definition of the non-compact form which admits the discrete series representations.

Representations of U$_q$SU$(1,1)$ at roots of unity have been studied in [21] and [28]. There are analogs of the unitary discrete series. This is an encouraging sign that their reps. might be useful in a definition of the q-deformed SU$(1,1)$ manifold relevant here, since the proposed CFT dual has discrete series representations of SU$(1,1)$.

The definition of the q-deformed AdS_3 used above and in sections 3.3 and 3.4 is based on U$_q$SU$(1,1)$, where the * operation satisfies the rel. (3.5) between the conjugation and the co-product. A different possibility which we have not explored much is to use U$_q$Sl$(2,\mathbb{R})$ [26, 29] since the classical isomorphism between SU$(1,1)$ and SL$(2,\mathbb{R})$ does not extend to the case $q \neq 1$. While the discussion above and in sections 3.3 and 3.4 finds evidence in favour of U$_q$SU$(1,1)$, it remains an interesting question to find analogous applications of U$_q$Sl$(2,\mathbb{R})$.

3.3 Discussion of descendants

While we focused on chiral primary operators (and their SU$(2) \times$ SU$(1,1)$ descendants) above, it is interesting to consider other operators which are obtained from these by acting with the superalgebra. We can use the notation of [6] to describe the set of reps.

There are reps. $D^{(l_1,l_2)}_{(E_0,s)}(n,k)$ The left and right SU(2) quantum numbers are $2J_L = (l_1 + l_2)$ and $2J_R = (l_1 - l_2)$. The left and right SU$(1,1)$ quantum numbers are given by $2h = (E_0 + s)$ and $2\bar{h} = (E_0 - s)$. The numbers (n,k) are quantum numbers under a symmetry group SO$(4) \times$ SO(n_T), where $n_T = 21$ for K_3, and $n_T = 5$ for T^4.

The oscillators $\alpha^{p,q}_{-n}$ have quantum numbers $(2J_L = n-1+p, 2J_R = n-1+q)$, and $2h_L = p+q+n-1$ and $2h_R = p-q+n-1$. This allows an identification of these α oscillators with figs. 1, 2, and 3 of [6]. The oscillator $\alpha^{(2,0)}_{-(l+2)}$ corresponds to $D^{(l+2,1)}_{(l+2,1)}$ of fig. 1. The oscillator $\alpha^{2,2}_{-(l+1)}$ corresponds to $D^{(l+2,0)}_{(l+2,0)}$ of fig. 2. The oscillators $\alpha^{(1,1)}_{-(l+2)}$ and $\alpha^{(0,0)}_{-(l+3)}$ are associated with $D^{(l+2,0)}_{(l+2,0)}$ of fig. 3.

The discussion in section 2 implies that there is a cutoff in the set of D's in fig. 1, given by $\max(l+2) = (N-2)$. For fig. 2 we have $\max(l+1) = (N-2)$. For fig. 3, we have $\max(l+2) = (N-1)$.

Once we have decided how the chiral primaries are cutoff, SUSY requires the associated descendants to be cutoff correspondingly. It is interesting to ask nevertheless how far we can predict the cutoffs on the descendants using only the quantum group symmetry SU$_q(1,1) \times$ SU$_q(2)$. For concreteness we discuss fig. 2. To carry out

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the argument we will need the fact both SU(2) and SU(1,1) are q-deformed. For SU(2) we have the usual $2J \leq (N-1)$, where $2J$ denotes the larger of $2J_L$ and $2J_R$. For $\mathrm{SU}_q(1,1)$ we have the cutoff $2h \leq N+1$, using the result of [28] on the consistent restriction to a set of unitary $\mathrm{SU}_q(1,1)$ reps.

Using these cutoffs we can correctly predict the cutoffs on the entire diamond of [6] except for the one rep. in the middle. The correct way to cut-off the set of reps is of course the one compatible with SUSY, and the extra state in the middle, allowed by unitarity of $\mathrm{SU}_q(1,1)$ and $\mathrm{SU}_q(2)$ should not appear. The consistent truncation should follow from constraints based on larger algebra. The simplest candidate is $\mathrm{SU}_q(2|1,1)$, but some larger algebra containing this as a subalgebra could be involved. It is very appealing nevertheless that the idea of q-deformation of the bosonic geometry alone is sufficiently powerful to predict the correct cutoffs for almost the entire figure.

3.4 Boundary of q-SU(1,1) and holography

We are arguing here that the spacetime gravity theory which can be reconstructed from the CFT, has finite N effects which can be understood in terms of a non-commutative spacetime. Among other things, we have argued that a non-commutative q-deformation of the SU(1,1) manifold is relevant. An important feature of the AdS-CFT correspondence, emphasized in [3] is that it involves a theory at the boundary being dual to the bulk gravity theory. In all the AdS/CFT examples of [1] the boundary theory lives on a space X which appears as a factor in a boundary space of the form $X \times Y$. Similarly in [30] we have string theory on a manifold which at infinity takes the form $S^3 \times S^2$ and is dual to a gauge theory on a manifold S^3. More generally we may expect X to appear as the base of a fibration at infinity. Another very simple string-gauge theory correspondence fits this general paradigm. The string theory of 2D QCD on a Riemann surface Σ [31, 32, 33] is related to a topological string theory on the cotangent bundle $T^*\Sigma$ [34]. The boundary of the R^2 fibre is S^1 so the boundary of the target is an S^1 fibration over Σ.

In this non-commutative context, an interesting property of the $q-\mathrm{SU}(1,1)$ manifold follows from the assumption that the standard holographic picture continues to hold. Assuming that holography continues to hold, the $q-\mathrm{SU}(1,1)$ manifold should have a boundary containing an ordinary commutative torus, on which the dual finite N orbifold CFT is defined. Indeed, one has [21] a tensor product structure in the space of unitary reps of $\mathrm{U}_q(\mathrm{SU}(1,1))$ of the form

$$\mathrm{U}_q(\mathrm{SU}(1,1)) = \mathrm{U}_q(\mathrm{SU}(2)) \times U(\mathrm{SU}(1,1))\,. \tag{3.7}$$

This means that if we define the non-commutative deformed algebra of functions on q-SU(1,1), using this class of unitary representations, we have a product structure on the q-deformed manifold:

$$\mathrm{SU}_q(1,1) = \mathrm{SU}(1,1) \times \mathrm{SU}_q(2)\,.$$

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allowing us to find the boundary of q-SU$(1,1)$. It is equal to the boundary of SU$(1,1)$ which is an ordinary torus times a compact q-manifold. (The factorization of the quantum geometry has been discussed in the context of Liouville theory in [35]). Consequently the q-deformation of the AdS space agrees with our expectations based on holography. The fact that the ordinary Riemann surface appears at the boundary allows us to hope that the precise expression of holography of the form expressed in [36] as

$$\left\langle exp \int \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0) \tag{3.8}$$

will have a q-generalization where Z_S is obtained from a theory on the non-commutative space $\mathrm{SU}_q(1,1)$, and the left hand side is computed from orbifold CFT on an ordinary Riemann surface.

The above discussion may raise a small puzzle which has a simple solution in this physical context. The derivation of the product structure (3.7) involves a somewhat unintuitive procedure of dropping the operators $X_{\pm}^{(N+1)}$ and adding back the operators $\frac{X_{\pm}^{(N+1)}}{[N+1]!}$ [21]. In such a discussion the full set of reps $V_{d,z}$ (in the notation of [28]) for arbitrary z plays an important role. For the discussion of particle excitations around the AdS background, a procedure which restricts attention (as for example in [28]) to the highest weights associated to $z = 1$ is more appropriate. As discussed in section 2, the X_+ operators are closely related to particle creation operators, and it makes sense to include the $\frac{X_{\pm}^{(N+1)}}{[N+1]!}$ if we allow ourselves to change the background, but not if we are discussing the physics around a fixed background as in the previous section. The freedom to add the extra operators should also be of help in setting up a spacetime description which allow accounting for the black hole entropy, a question which received an interesting discussion recently in [8]. For the purposes of discussing the global structure of spacetime, as we are doing here, it is natural to include the extra operators. Equivalently, the orbifold CFT contains the states relevant for black holes, so it should appear as the boundary of a quantum spacetime whose definition allows for operators which can create very heavy objects.

3.5 Nilpotence, finiteness and the q-deformation parameter

For quantum group at a root of unity, the representation ring truncates to a finite set. For $q = e^{i\pi/(k+2)}$ we get a restriction on the SU(2) spins $2j \leq k$. We saw that the cut-offs on the spectrum of generators $\alpha_{-l}^{(p,q)}$ of the chiral ring takes a simple form $2j \leq (N-1)$. The generators of the chiral ring are associated with single particle states in space-time and their cut-off is directly related to the q-deformation parameter associated with the geometrical deformation of $\mathrm{SU}(2) \times \mathrm{SU}(1,1)$ manifold to the non-commutative manifold $\mathrm{SU}_q(2) \times \mathrm{SU}_q(1,1)$. Earlier in section 2, we have argued for a q-deformation of the field oscillators and we derived in this context a deformed algebra $\mathrm{Sl}_q(2)$. This deformation was directly related to the nilpotence of the generators of the chiral ring. In both cases we find the same value of $q = e^{i\pi/(N+1)}$. While the truncation of the oscillator index can be interpreted simply in terms of the q-deformation of the geometry of background spacetime, the deformed Heisenberg algebras appear more directly related to the deformation of the dynamics of fluctuations around the background. In fact a number of physicists have explored the possibility of doing field theory in the non-commutative context by deforming the oscillator algebras of ordinary field theory, see for example [37]. The equality of the q-parameters obtained from these two different routes suggests that there are tight consistency conditions relating the deformation of space-time geometry and the deformation of field oscillator algebras.

An important role is played in non-commutative geometry of $\mathrm{SU}_q(2)$ as well as in deformed oscillator algebras by Hopf algebras. This leads to the suggestion that it might be possible to associate a Hopf algebra structure to the entire chiral ring and that such a structure will have an interesting role in characterizing the consistency conditions governing the deformed dynamics of fields on a deformed spacetime.

3.6 Towards a construction of non-commutative spacetime from the CFT

Non-commutative SU(2) manifold appears naturally in connection with SU(2) WZW CFT, which has been interpreted in terms of particle moving on $\mathrm{SU}(2)_q$ [38, 39]. An intuitive understanding of the q-deformation of S^3 in terms of an effective spacetime coordinate can be given. This would be related to the four dimensional coordinate of [40], given as

$$x^m = \bar{\Psi}\gamma^m\Psi \,, \tag{3.9}$$

which appeared in the calculations of the absorption cross-sections using the effective string model.

Consider the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ CFT symmetry currents built in terms of fermions as

$$J^A(z) = \Psi^+ \cdot t^A \cdot \Psi(z) \tag{3.10}$$

$$\bar{J}^A(\bar{z}) = \bar{\Psi}^+ \cdot t^A \cdot \bar{\Psi}(\bar{z}) \,. \tag{3.11}$$

Here we have the left and right moving Dirac components respectively. These can be written as linear combinations of the basic Majorana fermions

$$\psi_i^{a\alpha}(z) \,, \qquad \bar{\psi}_i^{\dot{a}\dot{\alpha}}(\bar{z}) \,. \tag{3.12}$$

As is now standard $a = 1, 2$ represent world sheet indices of the conformal field theory fermions and $\alpha, \dot{\alpha} = 1, 2$, the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ indices. We now consider a collective coordinate for the KK space. An effective coordinate of S^3 can be identified from the group element in a WZW type construction. In general

$$g(z, \bar{z}) = g_L(z) g_R(\bar{z}) \,.$$

We now use the well known fact that a global mode exists in the WZW theory. It is defined through the monodromy

$$V_q = g^{-1}(z)g(ze^{2\pi i})\,.$$

In terms of the current

$$V_q = Pe^{\oint J}\,.$$

It follows from the fact that the current J obeys an SU(2) Kac-Moody algebra that the global U_q degree of freedom obeys a q-deformed set of commutators. The appearance of a q-group structure is a typical phenomena in these theories and was studied extensively in the literature [38, 39]. For SU(2) the value of the deformation is given by $q = e^{i\pi/(k+2)}$ where k represents the level of the current algebra. In the present case the superconformal chiral algebra contains a an SU(2) current algebra with $k = N$ and

$$q = e^{\frac{i\pi}{N+2}}$$

is the related q-deformation parameter.

The $\mathrm{SU}(2)_q$ coordinate U_q is a conjugate to the monodromy V_q. In the discussion of SU(2) WZW one actually identifies the left and right monodromies. This is also appropriate for the present interpretation and identification of a center of mass coordinate on S_3. The requirement of the single-valuedness of the correlation functions of the chiral primaries leads to a correlation between the Monodromy Matrices on the left and the right. The coordinates U_q belong to the group $\mathrm{SU}(2)_q$ and the dynamics on this space is jut that of a q-deformed top.

To summarize starting with the SU(2) level $k = N$ algebra which appears as a subalgebra of the superconformal algebra an explicit matrix obeying the commutation relations of $\mathrm{SU}(2)_q$, with $q = e^{i\pi/(N+2)}$ provides a candidate for a non-commutative S^3 as part of the base space for the spacetime. This is not the same parameter that underlies the truncation of the generators of the chiral ring, or the deformed Heisenberg algebra we focused on in section 2. The operators of interest, the chiral primaries, have simple transformation properties under the SU(2) global symmetry of the CFT but do not necessarily have simple properties under the SU(2) current algebra (e.g. they do not have to be primaries of the SU(2) current algebra). This explains why their properties cannot be directly predicted from the current algebra. Nevertheless this line of argument should, with some modification to take into account the quantum effect of shifting k by 1, be useful in understanding the space-time coordinate. It is also worth noting that a consistent spacetime picture requires, as we argued, a q-deformation of the $\mathrm{SU}(1,1)$ part, whereas there is no manifest $\mathrm{SU}(1,1)$ current algebra in the orbifold CFT.

3.7 The AdS5 case

The conjecture that finite N effects are related to gravity on a non-commutative space has interesting consequences in the case of the duality between $AdS_5 \times S^5$ and $N = 4$

super-Yang Mills. As we saw earlier one of the simplest consequences of the non-commutativity is that we have to do KK reduction on a non-commutative sphere as opposed to a commutative one. Quantum groups also suggest a deformation of S^5 as $\mathrm{SU}_q(3)/\mathrm{SU}_q(2)$ [41] or perhaps a deformation starting from the quotient SO(6)/SO(5). KK reduction on such a space for $q \sim e^{i\pi/N}$ a root of unity will have the effect that there will be a truncation in the set of representations of SU(3) that appear. In the dual gauge theory this shows up as certain dependences between traces of powers of matrices ϕ^i. We have three complex matrices and an SU(3) action. The generating set of independent SU(N) invariant polynomials which can be made from these matrices will fall in a certain set of representations of SU(3) which will have N-dependent truncations. It will be interesting to see if these truncations are precisely the ones that appear for SU(3) for a q of the form $q \sim e^{i\pi/N}$.

4. Summary and outlook

We summarize here the main points developed above and outline some directions for further research.

The stringy exclusion principle on the space of chiral primaries implies a number of consequences for the spectrum of quantum supergravity on $AdS_3 \times S^3$. From the spacetime point of view there are qualitatively different aspects of this exclusion principle. The first involves a restriction on the number of particles of one kind. We related this to deformed Heisenberg algebras which we computed from the orbifold CFT. The second involves an upper bound on the number of generators of the chiral ring, which are related to single particle states in spacetime. We showed, using an explicit construction of the chiral primaries of the orbifold CFT, that this bound takes the simple form of a cutoff on the SU(2) spins of the chiral primary. This bound is stronger than the bound we get from general arguments based on the unitarity of the superconformal algebra. The precise cutoffs have a simple interpretation in the context of gravity on a non-commutative deformation of $S^3 \times AdS_3$ to $\mathrm{SU}_q(2) \times \mathrm{SU}_q(1,1)$. The hypothesis of a q-deformed spacetime with $q = e^{i\pi/(N+1)}$ is, therefore, more accurately predictive of the precise nature of the cutoffs on the generators of the ring of chiral primary operators coming from the S_N orbifold CFT, than general arguments based on the unitarity of the superconformal algebra, or on the existence of an SU(2) current algebra in the orbifold CFT with $k = N$.

Further work in the AdS_3 context is needed to understand the non-chiral primaries and the super-algebra descendants of the chiral primaries. The hypothesis of q-deformed spacetime explains a lot about the cutoffs on the superlagebra descendants of chiral primaries, and uses important features about the q-deformation of the non-compact part AdS_3. We expect that a q-deformation of $\mathrm{SU}(1,1|2)$, or perhaps an even larger algebra, will play an interesting role in improving the spacetime understanding of the cutoffs. Given our explicit construction of the chiral primaries, and the remark of [7] that the non-chiral primaries can be obtained from OPE's of the chiral primaries, we expect that the detailed form of the cutoffs on the non-chiral primaries can be deduced directly, by generalizing the arguments used here.

The properties of q-deformed AdS_3 also allowed a check that the hypothesis of q-deformed spacetime is consistent with holography. This suggests that the relation of [36] for correlators in a bulk theory with those of the boundary theory should admit a deformation to the context of a non-commutative spacetime.

Based on the idea that finite N effects are also related to a q-deformation of spacetime we outlined a conjecture on the finite N truncations of the spectrum of chiral primary operators in the case of the correspondence between type IIB supergravity on $AdS_5 \times S^5$ and $N = 4$ Yang Mills theory.

A recent paper of Witten shows that some finite N effects can be understood in terms of Chern-Simons actions. Given the relation between Chern Simons and quantum groups, there should be a close relation between the discussion of [36] and the remarks on finite N effects here. A clear elucidation of this would be interesting. It would require a better understanding of the relation between the truncations of the spectrum of chiral primaries studied here and the 't Hooft fluxes and Wilson loop properties studied in [36]. The investigation of the relation between Wilson loops and local operators of [42] should provide a start.

It remains to use the AdS/CFT correspondences to learn more about the theory of fluctuating geometries in the non-commutative setting. For example it would be very interesting to write an action for non-commutative gravity on the deformed $AdS_3 \times S^3$ space and test its properties using the dual CFT.

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